OLIMPIADA NAȚIONALĂ DE MATEMATICĂ
ETAPA LOCALĂ, 07 .02.2026
CLASA a XI-a
Barem de notare

Subiectul I (21 puncte)

Fie mulțimea $M = \{X \in \mathcal{M}_2(\mathbb{C}) | \exists k \in \mathbb{N}, k \geq 2, X^k = O_2\}$.

a) Arătați că dacă $X \in M \Rightarrow X^2 = O_2$.
b) Calculați $\det(I_2 + A + A^2 + \ldots + A^{2026})$ unde $A \in M$.
c) Studiați surjectivitatea funcției $f : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C}), f(X) = X^{2026}$.

Soluție:

a) Fie $X \in M \Rightarrow X^k = O_2 \Rightarrow \det(X) = 0 \Rightarrow$ **(3 puncte)**
$\Rightarrow X^2 = tr(X) \cdot X \Rightarrow X^k = \big(tr(X)\big)^{k-2} \cdot X^2 \Rightarrow \big(tr(X)\big)^{k-2} \cdot X^2 = O_2 \Rightarrow X^2 = O_2$ **(4 puncte)**

b) Fie $A \in M \Rightarrow A^2 = O_2 \Rightarrow I_2 + A + A^2 + \ldots + A^{2026} = I_2 + A \Rightarrow \det(I_2 + A + A^2 + \ldots + A^{2026}) = \det(I_2 + A) = 1 + \det(A) + tr(I_2) \cdot tr(A) - tr(I_2 \cdot A) = \det(I_2) = 1$ **(7 puncte)**
Soluție alternativă: Fie $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, A \in M \Rightarrow A^2 = O_2 \Rightarrow A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, a^2 + bc = 0$
$\det(I_2 + A + A^2 + \ldots + A^{2026}) = \det(I_2 + A) = \begin{vmatrix} 1+a & b \\ c & 1-a \end{vmatrix} = 1 - a^2 - bc = 1$

c) Fie $A = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} \Rightarrow A^2 = O_2 \Rightarrow A \in M \subset \mathcal{M}_2(\mathbb{C})$
$f(X) = A \Rightarrow X^{2026} = A \,/\,()^2 \Rightarrow X^{2 \cdot 2026} = O_2 \Rightarrow X \in M \Rightarrow X^2 = O_2 \Rightarrow X^{2026} = O_2 \Rightarrow A = O_2$
(Contradicție) $\Rightarrow f$ nu este surjectivă. **(7 puncte)**

Subiectul II (21 puncte)

Se consideră șirul $(x_n)_{n \geq 1}$ cu termenul general $x_n = \sum_{k=1}^{n} \frac{n}{n^2+n+k}$.

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Calculați $\lim\limits_{n \to \infty} x_n$ și $\lim\limits_{n \to \infty} n(1 - x_n)$.

Soluție:

$\frac{n^2}{n^2+2n} \leq \sum_{k=1}^{n} \frac{n}{n^2+n+k} \leq \frac{n^2}{n^2+n+1}$. Cum $\lim\limits_{n \to \infty} \frac{n^2}{n^2+2n} = \lim\limits_{n \to \infty} \frac{n^2}{n^2+n+1} = 1 \Rightarrow \lim\limits_{n \to \infty} x_n = 1$ **(7 puncte)**

$n(1 - x_n) = n\left(1 - \sum_{k=1}^{n} \frac{n}{n^2+n+k}\right) = n \sum_{k=1}^{n} \left(\frac{1}{n} - \frac{n}{n^2+n+k}\right) = \sum_{k=1}^{n} \frac{n+k}{n^2+n+k}$ **(7 puncte)**

$\sum_{k=1}^{n} \frac{n+k}{n^2+2n} \leq \sum_{k=1}^{n} \frac{n+k}{n^2+n+k} \leq \sum_{k=1}^{n} \frac{n+k}{n^2+n+1} \Leftrightarrow \frac{3n^2+n}{2n^2+4n} \leq \sum_{k=1}^{n} \frac{n+k}{n^2+n+k} \leq \frac{3n^2+n}{2n^2+2n+2}$.

Cum $\lim\limits_{n \to \infty} \frac{3n^2+n}{2n^2+4n} = \lim\limits_{n \to \infty} \frac{3n^2+n}{2n^2+2n+2} = \frac{3}{2} \Rightarrow \lim\limits_{n \to \infty} n(1 - x_n) = \frac{3}{2}$ **(7 puncte)**

Subiectul III (21 puncte)

a) Fie $A, B \in \mathcal{M}_2(\mathbb{R})$ astfel încât $AB = BA$. Arătați că $det(A^2 + B^2) \geq 0$
b) Dacă $X \in \mathcal{M}_2(\mathbb{R})$ și $det(X) > 0$ arătați că $det(X - I_n + X^{-1}) \geq 0$

Soluție:

a) Fie $A, B \in \mathcal{M}_2(\mathbb{R})$ astfel încât $AB = BA \Rightarrow A^2 + B^2 = (A + iB)(A - iB) \Rightarrow$
$det(A^2 + B^2) = \det(A + iB)\,(A - iB) = det(A + iB) \cdot det(A - iB) = det(A + iB) \cdot \overline{det(A + iB)} =$
$= |det(A + iB)|^2 \geq 0$ **(7 puncte)**

b) Fie $X \in \mathcal{M}_2(\mathbb{R})$ cu $det(X) > 0 \Rightarrow X - I_n + X^{-1} = X^{-1}(X^2 - X + I_n)$ **(4 puncte)**
$X \cdot X^{-1} = I_n \Rightarrow det(X) \cdot det(X^{-1}) = 1 \Rightarrow det(X^{-1}) = \frac{1}{det(X)} > 0$ **(3 puncte)**
$det(X - I_n + X^{-1}) = det\big(X^{-1}(X^2 - X + I_n)\big) = det\,(X^{-1}) \cdot det\,((X^2 - X + I_n) =$
$= det\,(X^{-1}) \cdot\ det\,\left((X - \frac{1}{2}I_n)^2 + (\frac{\sqrt{3}}{2}I_n)^2\right) \geq 0$ **(7 puncte)**

Subiectul IV (21 puncte)

Fie un șir de numere reale $(a_n)_{n\geq 1}$, pentru care $a_1 > 1$ și $a_{n+1} = \sqrt{a_n}\big(1 + \sqrt{a_n}\big) - 1$ pentru orice $n \in \mathbb{N}^*$. Calculați $\lim\limits_{n\to\infty} \frac{\sqrt{a_n}}{n}$ și $\lim\limits_{n\to\infty} \frac{n^2 - 4\cdot a_n}{n\cdot \ln n}$.

Soluție:

- $a_n > 1,\ \forall\, n \in \mathbb{N}^*$ **(2 puncte)**
- $a_{n+1} = \sqrt{a_n}\big(1 + \sqrt{a_n}\big) - 1 \Rightarrow a_{n+1} - a_n = \sqrt{a_n} - 1 > 0 \Rightarrow (a_n)_{n\geq 1}$ strict crescător.
  Presupunem că $(a_n)_{n\geq 1}$ este mărginit $\Rightarrow (a_n)_{n\geq 1}$ convergent $\Rightarrow \lim\limits_{n\to\infty} a_n = l \in \mathbb{R} \Rightarrow$ $l = \sqrt{l} + l - 1 \Rightarrow l = 1$ imposibil $\Rightarrow \lim\limits_{n\to\infty} a_n = \infty$ **(3 puncte)**
- $\frac{a_{n+1}}{a_n} = \frac{1}{\sqrt{a_n}} + 1 - \frac{1}{a_n} \Rightarrow \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ **(1 punct)**
- $\lim\limits_{n\to\infty} \frac{\sqrt{a_n}}{n} \overset{Stolz-Cesaro}{\Longrightarrow} \lim\limits_{n\to\infty} \left(\sqrt{a_{n+1}} - \sqrt{a_n}\right) = \lim\limits_{n\to\infty} \frac{a_{n+1} - a_n}{\sqrt{a_{n+1}} + \sqrt{a_n}} = \lim\limits_{n\to\infty} \frac{\sqrt{a_n} - 1}{\sqrt{a_{n+1}} + \sqrt{a_n}} =$
  $\lim\limits_{n\to\infty} \frac{\sqrt{a_n}\left(1 - \frac{1}{\sqrt{a_n}}\right)}{\sqrt{a_n}\left(\sqrt{\frac{a_{n+1}}{a_n}} + 1\right)} = \frac{1}{2}$ **(5 puncte)**
- $\lim\limits_{n\to\infty} \frac{n^2 - 4\cdot a_n}{n\cdot \ln n} = \lim\limits_{n\to\infty} \frac{(n - 2\sqrt{a_n})(n + 2\sqrt{a_n})}{n\cdot \ln n} = \lim\limits_{n\to\infty} \frac{n - 2\sqrt{a_n}}{\ln n} \cdot \frac{n + 2\sqrt{a_n}}{n} =$ $= \lim\limits_{n\to\infty} \frac{n - 2\sqrt{a_n}}{\ln n} \cdot (1 + 2\frac{\sqrt{a_n}}{n}) = L$ **(3 puncte)**
- $\lim\limits_{n\to\infty} \frac{n - 2\sqrt{a_n}}{\ln n} \overset{Stolz-Cesaro}{\Longrightarrow} \lim\limits_{n\to\infty} \frac{1 + 2\sqrt{a_n} - 2\sqrt{a_{n+1}}}{\ln(n+1) - \ln n} = \lim\limits_{n\to\infty} \frac{n\cdot\left(1 - 2\left(\sqrt{a_{n+1}} - \sqrt{a_n}\right)\right)}{ln\left(1 + \frac{1}{n}\right)^n} =$
  $= \lim\limits_{n\to\infty} \frac{n\cdot\left(1 - 2\frac{a_{n+1} - a_n}{\sqrt{a_{n+1}} + \sqrt{a_n}}\right)}{ln\left(1 + \frac{1}{n}\right)^n} = \lim\limits_{n\to\infty} \frac{n\cdot\left(1 - 2\frac{\sqrt{a_n} - 1}{\sqrt{a_{n+1}} + \sqrt{a_n}}\right)}{ln\left(1 + \frac{1}{n}\right)^n} = \lim\limits_{n\to\infty} \frac{n\cdot\left(\sqrt{a_{n+1}} + \sqrt{a_n} - 2\sqrt{a_n} + 2\right)}{ln\left(1 + \frac{1}{n}\right)^n\left(\sqrt{a_{n+1}} + \sqrt{a_n}\right)} =$
  $\lim\limits_{n\to\infty} \frac{n\cdot\left(\sqrt{a_{n+1}} - \left(\sqrt{a_n} - 2\right)\right)}{ln\left(1 + \frac{1}{n}\right)^n\left(\sqrt{a_{n+1}} + \sqrt{a_n}\right)} = \lim\limits_{n\to\infty} \frac{n\cdot\left(a_{n+1} - \left(\sqrt{a_n} - 2\right)^2\right)}{ln\left(1 + \frac{1}{n}\right)^n\left(\sqrt{a_{n+1}} + \sqrt{a_n}\right)\left(\sqrt{a_{n+1}} + \sqrt{a_n} - 2\right)} =$
  $\lim\limits_{n\to\infty} \frac{n\cdot\left(a_{n+1} - a_n + 4\sqrt{a_n} - 4\right)}{ln\left(1 + \frac{1}{n}\right)^n\left(\sqrt{a_{n+1}} + \sqrt{a_n}\right)\left(\sqrt{a_{n+1}} + \sqrt{a_n} - 2\right)} = \lim\limits_{n\to\infty} \frac{n\cdot\left(\sqrt{a_n} - 1 + 4\sqrt{a_n} - 4\right)}{ln\left(1 + \frac{1}{n}\right)^n\left(\sqrt{a_{n+1}} + \sqrt{a_n}\right)\left(\sqrt{a_{n+1}} + \sqrt{a_n} - 2\right)} =$
  $\lim\limits_{n\to\infty} \frac{5n\cdot\left(\sqrt{a_n} - 1\right)}{ln\left(1 + \frac{1}{n}\right)^n\left(\sqrt{a_{n+1}} + \sqrt{a_n}\right)\left(\sqrt{a_{n+1}} + \sqrt{a_n} - 2\right)} =$
  $\lim\limits_{n\to\infty} \frac{5n^2\cdot\left(\frac{\sqrt{a_n}}{n} - \frac{1}{n}\right)}{ln\left(1 + \frac{1}{n}\right)^n n^2\left(\frac{\sqrt{a_{n+1}}}{n+1}\left(1 + \frac{1}{n}\right) + \frac{\sqrt{a_n}}{n}\right)\left(\frac{\sqrt{a_{n+1}}}{n+1}\left(1 + \frac{1}{n}\right) + \frac{\sqrt{a_n}}{n} - \frac{2}{n}\right)} = \frac{5}{2}. \Rightarrow L = \frac{5}{2}\cdot 2 = 5$ **(7 puncte)**.